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Mixed convection heat transfer from a vertical plate to non-Newtonian fluids

T.-Y. Wang
Department of Mechnical Engineering, Mingchi Institute of Technology, Taishan, Taipei, Taiwan, Republic of China

The nonsimilar boundary-layer analysis of steady laminar mixed-convection heat transfer between a vertical plate and non-Newtonian fluids is extended and unified. A mixed-convection parameter ζ is proposed to replace the conventional Richardson number, $Gr/\mathrm{Re}^{2/(2-n)}$ and to serve as a controlling parameter that determines the relative importance of the forced and the free convection. The value of mixed-convection parameter lies between 0 and 1. In addition, the power-law model is used for nonNewtonian fluids with exponent $n < 1$ for pseudoplastics; $n = 1$ for Newtonian fluids; and $n > 1$ for dilatant fluids. Furthermore, the coordinates and dependent variables are transformed to yield computationally efficient numerical solutions that are valid over the entire range of mixed convection, from the pure forced-convection limit to the pure free-convection limit, and the whole domain of non-Newtonian fluids, from pseudoplastics to dilatant fluids. The effects of the mixed-convection parameter, the power-law viscosity index, and the generalized Prandtl number on the velocity profiles, the temperature profiles, as well as on the wall skin friction and heat transfer rate are clearly illustrated for both cases of buoyancy assisting and opposing flow conditions.

Keywords: mixed-convection parameter; non-Newtonian fluids; power-law viscosity index; wall skin friction; heat transfer rate

Introduction

The analysis of convective heat transfer of non-Newtonian fluids over a vertical plate is important for the thermal design of industrial equipment dealing with molten plastics, polymeric liquids, foodstuffs, or slurries. Thus, considerable efforts have been directed toward major aspects of this coupled, nonlinear boundary-layer problem. For example, Acrivos et al. (1960) studied forced-convection heat transfer from external surface to nonNewtonian fluids of infinitely large Prandtl numbers. Lin and Shih (1980) devised a local similarity technique for investigating combined free- and forced-convection from a vertical plate to power-law fluids. Kim et al. (1983) applied Merk's series expansion method to solve the forced convection between two-dimensional (2-D) or axisymmetric bodies and non-Newtonian fluids. Huang and Chen (1984) presented a numerical analysis for forced convection over a flat plate in power-law fluids. Nakayama et al. (1986) used integral method to obtain the solution of forced-convection heat transfer from external surfaces immersed in non-Newtonian fluids. Kleinstreuer and Wang (1988) and Wang and Kleinstreuer (1988, 1990) developed new transformation parameters for analyzing thermal convection of non-Newtonian fluids past vertical slender cylinders, rotating spheres, and axisymmetric bodies. Recently, Huang and Lin (1993) selected the buoyancy effect as the controlling parameter and introduced a generalized Prandtl number as the streamwise parameter to solve the nonlinear problem of mixed-convection from a vertical plate to power-law fluids. More recently, Wang (1993) gave an analysis of steady Laminar mixed-convection heat transfer between a horizontal plate and non-Newtonian fluids. All of the previous investigations mentioned above are valid only for pure forced convection, pure free convection, or mixed convection for only a limited range of Richardson numbers and somewhat restrictive series expansions; local similarity techniques or integral methods were employed to solve particular boundary-layer equations.

In the present analysis of mixed convection from a vertical plate to non-Newtonian fluids, a mixed-convection parameter is proposed, and a powerful coordinate transformation is introduced for the nondimensionalization of the governing equations. The resulting nonsimilar equations are solved by using a very effective finite difference scheme to give solutions that are uniformly valid over the entire range of mixed convection intensity from pure forced-convection limit ($\zeta = 0$) to pure free-convection limit ($\zeta = 1$) and the whole domain of non-Newtonian fluids from pseudoplastics ($n < 1$) to dilatant fluids ($n > 1$).

Analysis

Consider steady laminar boundary-layer flow of a non-Newtonian fluid past a vertical plate at uniform wall temperature. The fluid properties are considered to be constant, except that the density variations within the fluid are allowed to contribute to the buoyancy force. When the wall is higher than the ambient temperature; i.e., $T_w > T_\infty$ or $Z = 1$, the buoyancy force will aid the upwardly directed uniform stream, and when $T_w < T_\infty$, or $Z = -1$, the resulting buoyancy force

Address reprint requests to Professor Wang at the Department of Mechanical Engineering, Mingchi Institute of Technology, 84 Gungjuan Road, Taishan, Taipei 24306 Taiwan, ROC.

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will retard the forced flow. By employing the Boussinesq approximations and making use of the power-law viscosity model, the governing equations for the problem under consideration are as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=Zg\beta(T-T_\infty)+\frac{K}{\rho}\frac{\partial}{\partial y}\left[\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right] \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The associated boundary conditions are as follows:

$$\text{at } y=0:\quad u=0,\ v=0 \text{ and } T=T_w=\text{constant} \tag{4a}$$

$$\text{as } y\to\infty:\quad u=u_\infty \text{ and } T=T_\infty \tag{4b}$$

The system equations can be reduced by using the stream function approach:

$$u=\frac{\partial\Psi}{\partial y} \tag{5a}$$

and

$$v=-\frac{\partial\Psi}{\partial x} \tag{5b}$$

together with the transformations

$$\xi=x/L;\ \eta=\lambda\xi^{-1/(n+1)}y/L \tag{6, 7}$$

$$\Psi=\alpha\lambda\xi^{1/(n+1)}F(\xi,n) \tag{8}$$

$$\theta(\eta)=(T-T_\infty)/(T_w-T_\infty) \tag{9}$$

The dimensionless buoyancy parameter λ is defined as follows:

$$\lambda=\mathrm{Re}^{1/(n+1)}+\mathrm{Ra}^{1/(3n+1)}=\mathrm{Re}^{1/(n+1)}/(1-\zeta)=\mathrm{Ra}^{1/(3n+1)}/\zeta \tag{10}$$

where $\mathrm{Re}=\rho u_\infty^{2-n}L^n/K$ and $\mathrm{Ra}=\rho g\beta|T_w-T_\infty|L^{2n+1}/(k\alpha^n)$ are the generalized Reynolds number and the generalized Raleigh number, respectively, with L being the characteristic length of the plate. The mixed-convection parameter ζ, covering the entire domain of mixed convection from the pure forced convection ($\zeta=0$) to the pure free convection ($\zeta=1$), is defined as follows:

$$\zeta=\mathrm{Ra}^{1/(3n+1)}/[\mathrm{Re}^{1/(n+1)}+\mathrm{Ra}^{1/(3n+1)}] \tag{11}$$

As a result, the continuity equation is automatically satisfied, and the momentum and energy equations are transformed to the following:

$$\mathrm{Pr}^{2-n}(|F''|^{n-1}F'')'+1/(n+1)FF''+Z\mathrm{Pr}^{2-n}\xi\zeta^{3n+1}\theta=\xi\left[F'\frac{\partial F'}{\partial\xi}-F''\frac{\partial F}{\partial\xi}\right] \tag{12}$$

and

$$\xi^{(n-1)/(n+1)}\theta''+\frac{1}{n+1}F\theta'=\xi\left[F'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial F}{\partial\xi}\right] \tag{13}$$

where Pr is the generalized Prandtl number:

$$\mathrm{Pr}=\left(\frac{K}{\rho}\right)^{1/(2-n)}L^{(2-2n)/(2-n)}\lambda^{3(n-1)/(2-n)}/\alpha \tag{14}$$

The corresponding boundary conditions are as follows

$$F(\xi,0)=F'(\xi,0)=0 \text{ and } \theta(\xi,0)=1 \tag{15a}$$

$$F'(\xi,\infty)=\mathrm{Pr}\,(1-\zeta)^{(n+1)/(2-n)} \text{ and } \theta(\xi,\infty)=0 \tag{15b}$$

In the foregoing equations, the primes denote differentiation with respect to η.

The physical quantities of primary interest are the local skin friction coefficient C_f and the local Nusselt number Nu. With the definition of the local skin friction coefficient $C_f=2\tau_w/(\rho u_\infty^2)$, a dimensionless skin friction group, SFG, can be formed as follows:

$$\mathrm{SFG}=1/2C_f\,\mathrm{Re}_x^{1/(n+1)}=\mathrm{Pr}^{-n}(1-\zeta)^{3n/(n-2)}[F''(\xi,0)]^n \tag{16}$$

where Re_x is the generalized Reynolds number based on x

$$\mathrm{Re}_x=\rho u_\infty^{2-n}x^n/K \tag{17}$$

Similarly, with the local Nusselt number based on the length of plate, $\mathrm{Nu}=hL/k$, a dimensionless heat transfer parameter

Notation

C_f	local skin friction
F	dimensionless stream function
g	gravitational acceleration
h	local heat transfer coefficient
k	thermal conductivity
K	fluid consistency index for power-law fluid
L	length of the plate
Nu	Nusselt number based on L
Nu_x	Nusselt number based on x
n	power-law viscosity index
Pr	generalized Prandtl number
q	local heat transfer rate
Ra	generalized Rayleigh number
Re	generalized Reynolds number based on L
Re_x	generalized Reynolds number based on x
T	temperature
u	velocity component in the x-direction
v	velocity component in the y-direction
x	streamwise coordinate
y	coordinate normal to the surface
Z	dimensionless parameter; $Z=1$ for heated plates, and $Z=-1$ for cooled plates

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient
θ	dimensionless temperature
ρ	density of fluid
ζ	mixed-convection parameter
Ψ	stream function
λ	buoyancy parameter
τ	shear stress
η	dimensionless coordinate
ξ	dimensionless coordinate

Subscripts

e	boundary-layer edge condition
w	wall condition
∞	ambient condition

Table 1 Comparison of $Nu_x/\sqrt{Re_x^*}$ for mixed convection between a vertical flat plate and Newtonian fluids

Gr_x/Re_x^{*2}	Pr = 0.72		Pr = 10		Pr = 100	
	Lloyd & Sparrow (1970)	Present method	Lloyd & Sparrow (1970)	Present method	Lloyd & Sparrow (1970)	Present method
0.0	0.2956	0.29550	0.7281	0.72768	1.572	1.5708
0.01	0.2979	0.29801	0.7313	0.73166	1.575	1.5754
0.04	0.3044	0.30502	0.7404	0.74315	1.585	1.5891
0.1	0.3158	0.31732	0.7574	0.76431	1.606	1.6152
0.4	0.3561	0.36097	0.8259	0.84707	1.691	1.7273
1.0	0.4058	0.41441	0.9212	0.95823	1.826	1.8959

$Gr_x = g\beta(T_w - T_\infty)x^3/v^2$; $Re_x^* = u_\infty x/v$

Table 2 Comparison of $1/2C_fRe_x^{1/(n+1)}$ for forced convection between a flat plate and non-Newtonian fluids

n	Acrivos et al. (1960)	Kim et al. (1983)	Huang & Chen (1984)	Present method
0.5	0.5755	—	0.5808	0.5761
0.8	—	—	0.4054	0.4052
1.0	0.3320	—	0.3320	0.3320
1.2	0.2750	0.2780	0.2780	0.2778
1.5	0.2189	—	0.2203	0.2201

HTP can be defined as follows:

$$HTP = Nu/\lambda = -\xi^{-1/(n+1)}\,\theta'(\xi, 0) \tag{18}$$

which is valid for the whole region of mixed convection. Whereas, the traditional heat transfer group HTG, which restricted to $0 \le \zeta < 1$, is given as follows:

$$HTG = Nu_x\, Re_x^{-1/(n+1)} = -(1-\zeta)^{-1}\theta'(\xi, 0) \tag{19}$$

with the local Nusselt number based on x, $Nu_x = hx/k$.

Numerical solution

The transformation of the governing equations reduces the numerical work significantly. The resulting system of coupled nonlinear equations and associated boundary conditions is solved by an implicit finite difference technique. This technique is a modified version of the described in Cebeci and Bradshaw (1984).

The 2-D grid is nonuniform to accommodate the steep velocity and temperature at the wall, particularly in the vicinity of the singular point at $\xi = 0$; i.e., the leading edge of the vertical plate. The location of the boundary-layer edge η_∞ depends strongly on the power-law viscosity index n; the mixed-convection parameter ζ; and the fluid Prandtl number Pr. for example, $\eta_\infty(n = 1.0,\ \zeta = 0,\ Pr = 10) \approx 8$ and η_∞ $(n = 0.5,\ \zeta = 1.0,\ Pr = 10) \approx 40$. Numerical error testing has been accomplished by straightforward repeat calculations with finer meshes to check grid independence of the results and by local mesh refinement in the η-direction with smooth transition to the coarser region.

Results and discussion

The numerical computations were carried out for the entire range of mixed thermal convection. Of special interest are the effects of the mixed-convection parameter as well as the power-law viscosity index and the generalized Prandtl number on the local skin friction coefficient and the local Nusselt number. Before the results of the parametric sensitivity analyses are shown, the accuracy of the present computer simulation model is examined.

Because measured datasets or analytical/numerical solutions are not available to check the accuracy of the present system, the computer simulation results of the present study for special cases are compared with datasets published in the open literature. In Table 1, the local Nusselt numbers for Newtonian fluids subjected to mixed convection along a vertical flat plate using the restrictive Richardson number, Gr_x/Re_x^{*2}, as a parameter are compared to the results obtained by Lloyd and Sparrow (1970). The agreement is very good when $Gr_x/Re_x^{*2} \le 0.1$. However, for larger Gr_x/Re_x^{*2} values, the results of the local similarity solutions become somewhat erroneous. Tables 2 and 3, respectively, contain wall skin friction and heat transfer comparison for forced convection between a flat plate and non-Newtonian fluids. It can be seen from these tables that the datasets match very well.

The streamwise velocity $u(x, y)$ is expressed in terms of $F'(\xi, \eta)$ as $u = (\alpha/L)\lambda^2 F'(\xi, \eta)$. The evolution of the profiles of $F'(\xi, \eta)$ from pure forced-convection limit ($\zeta = 0$) to pure free-convection limit ($\zeta = 1$) for buoyancy assisting case ($Z = 1$) is presented in Figures 1(a) and 1(b) for $n = 0.5$ and 1.5, respectively. The transition of the profiles from one limit to the other is clearly shown in these figures.

Table 3 Comparison of $Nu_x\,Re_x^{-1/(n+1)}$ for forced convection between a flat plate and non-Newtonian fluids

n	ξ	Huang & Chen (1984)	Present method
	0.01	0.4170	0.4143
0.5	0.1	0.5426	0.5386
	1.0	0.7034	0.6986
	0.01	1.0544	1.0454
1.5	0.1	0.8967	0.8962
	1.0	0.7683	0.7677

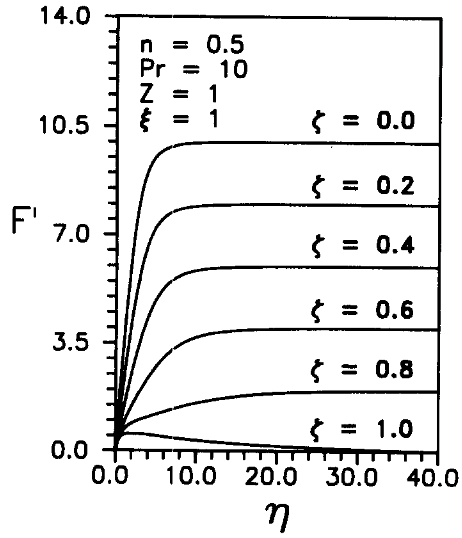

Figure 1(a) $F(\xi, \eta)$ profiles on a vertical plate for pseudoplastics covering the entire free-forced convection region

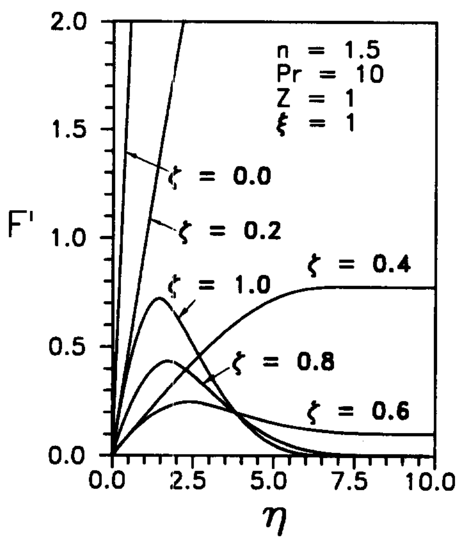

Figure 1(b) $F(\xi, \eta)$ profiles on a vertical plate for dilatant fluids covering the entire free-forced convection region

Figure 2 shows $F'(\xi, \eta)$ profiles at different locations along the vertical flat plate for opposing flow ($Z = -1$). The buoyancy force, similar to an adverse pressure gradient, retards forced convection, and typical S-shaped velocity profiles may occur. As indicated in Figure 2, for this particular set of system parameters, flow separation is eminent for $\xi > 0.908$ that is preceded for $(\partial u/\partial \eta) = 0$ at $\eta = 0$.

The influence of the generalized Prandtl number on the $F'(\xi, \eta)$ profiles is depicted in Figure 3 for pure free convection ($\zeta = 1$) of dilatant fluids over a vertical flat plate at $\xi = 1$; it can be seen that the dimensionless velocity increases with increasing Pr. Further examination of Figure 3 reveals that the wall gradients, $dF'/d\eta$ at $\eta = 0$, are nearly constant for three different generalized Prandtl numbers.

The representative dimensionless temperature profiles within the thermal boundary layer at $\xi = 1$ are respectively given in Figures 4(a) and 4(b) for $n = 0.5$ and 1.5. It is found that the dimensionless temperature gradients at the wall decrease monotonically with increasing buoyancy force for pseudoplastics. However, this is not true for dilatant fluids. The reason is that the dimensionless coordinate η is related to the buoyancy force; i.e., $\eta \approx (1 - \zeta)^{-1}$ for forced-convection dominated regime and $\eta \approx \zeta^{-1}$ for free-convection dominated regime, and the effect strongly influences the dimensionless temperature distributions for dilatant fluids.

The variations of $\theta(\xi, \eta)$ with the generalized Prandtl number and the power-law viscosity index are illustrated in Figure 5 for $\zeta = 0.5$ and $\xi = 1$. The dimensionless temperature gradient at the wall is increasing with a decrease in power-law viscosity index or an increase in generalized Prandtl number.

The effect of the mixed-convection parameter on SFG are demonstrated in Figure 6 for pseudoplastics. As shown in this figure, for any value of ξ, the wall skin friction group increases with increasing mixed-convection parameter for assisting flow. On the contrary, for a given value of ξ, the wall skin friction group decreases with increasing mixed-convection parameter for opposing flow.

Figures 7(a) and 7(b) present the distributions of the heat transfer parameter HTP for pseudoplastics and dilatant fluids, respectively. The proper selection of dimensionless buoyancy parameter λ allows the simulation of HTP for the entire

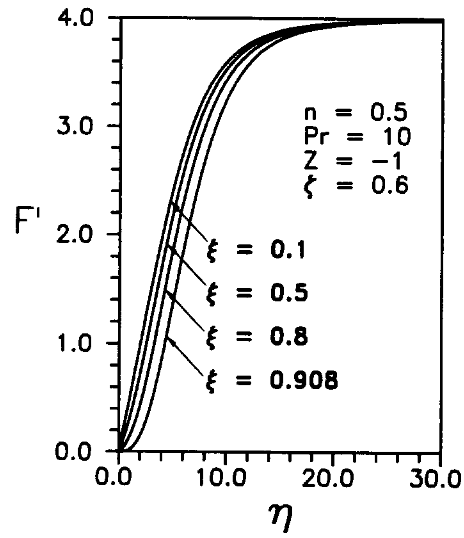

Figure 2 $F(\xi, \eta)$ profiles for buoyancy opposing opposing flow of pseudoplastics over a vertical flat plate

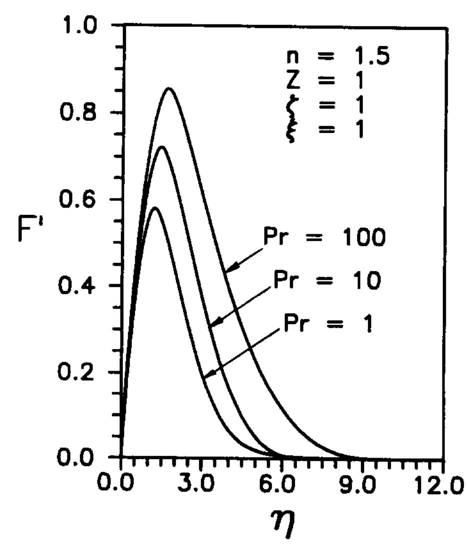

Figure 3 The influence of generalized Prandtl number on the $F(\xi, \eta)$ profiles of free convection flow for dilatant fluids

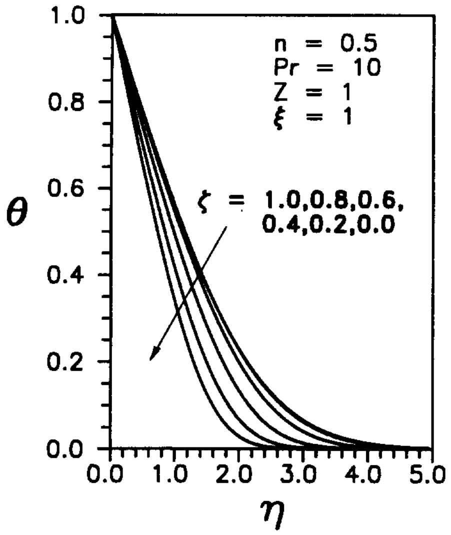

Figure 4(a) Dimensionless temperature profiles on a vertical plate for pseudoplastics covering the entire free-forced convection region

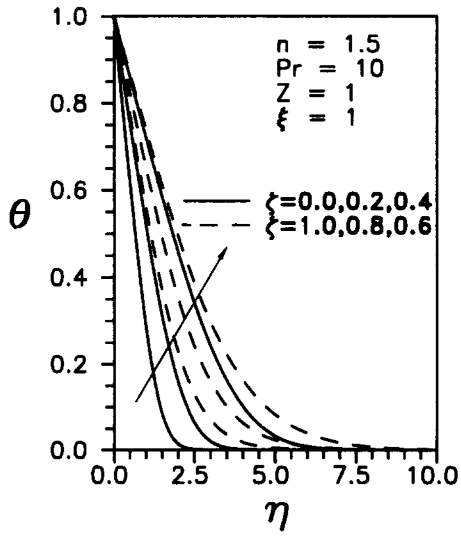

Figure 4(b) Dimensionless temperature profiles on a vertical plate for dilatant fluids covering the entire free-forced convection region

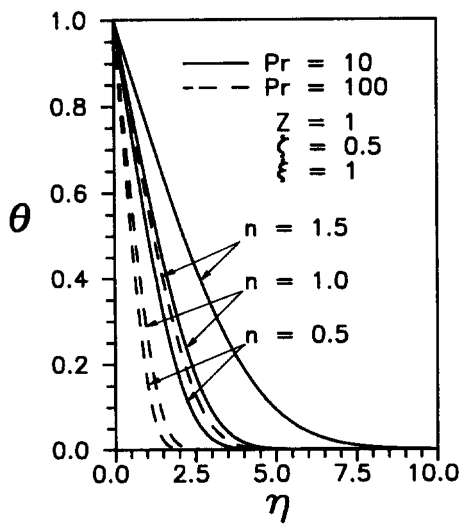

Figure 5 The variations of $\theta(\xi, \eta)$ with the generalized Prandtl number and power-law viscosity index

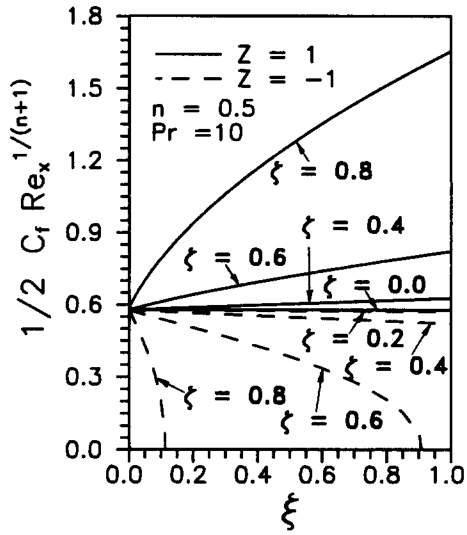

Figure 6 The effect of mixed-convection parameter on skin friction group (SFG)

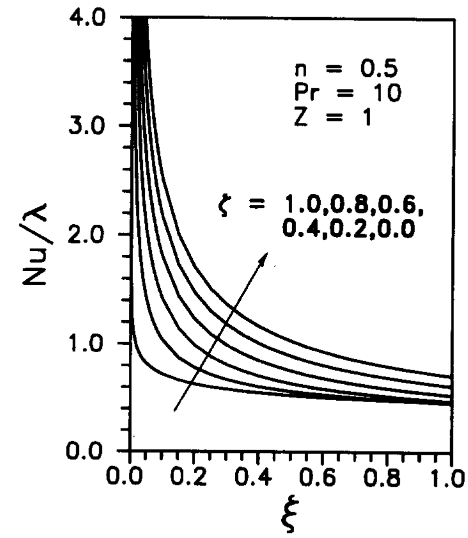

Figure 7(a) The distribution of the heat transfer parameter (HTP) for pseudoplastics covering the entire free-forced convection region

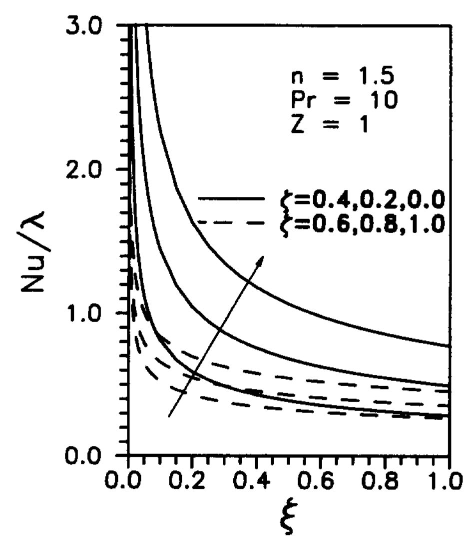

Figure 7(b) The distribution of the HTP for dilatant fluids covering the entire free-forced convection region

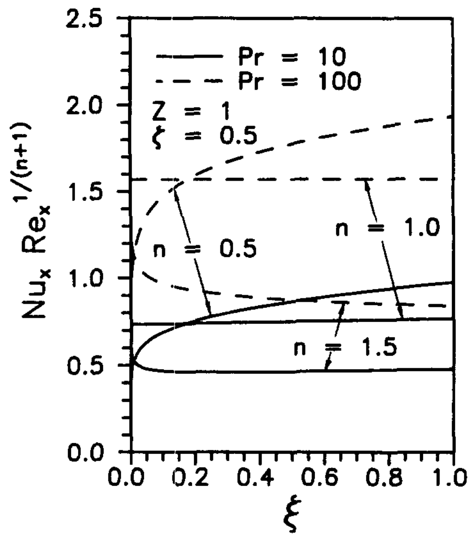

Figure 8 The effects of the generalized Prandtl number and the power-law viscosity index on heat transfer group (HTG)

mixed-convection range. The effects of the generalized Prandtl number and the power-law viscosity index on heat transfer group (HTG) are shown in Figure 8 for a fixed mixed-convection parameter. From this figure, it can be seen that the rate of heat transfer increases with increasing generalized Prandtl number for any power-law viscosity index n. Furthermore, for a given generalized Prandtl number; $Nu_x\, Re_x^{-1/(n+1)}$ increases with increasing ξ as $n < 1$ and decreases with increasing ξ as $n > 1$.

Conclusions

A generalized analysis of laminar mixed-convection heat transfer between non-Newtonian fluids and a vertical flat plate has been presented. A mixed-convection parameter is proposed to replace the conventional Richardson number. By introducing this new mixed-convection parameter to scale the relative contribution of the forced and free convection properly, a set of nonsimilar equations was obtained that provides for very accurate finite difference solutions over the whole range of mixed-convection flows from the pure forced-convection limit to the pure free-convection limit.

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